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SUMMARY

ASYMPTOTIC EVALUATIONS IN ANALYTICAL THEORY OF NUMBERS

PH.D. Advisor
Prof. univ. dr. **Dumitru Popa**

Ph.D. Student
Magdalena Corciovei (Bănescu)

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Contents

Preface

This paper designed as a thesis to obtain his Ph.D. in mathematics, is a brief foray into the analytic theory of numbers, specifically in the field of asymptotic evaluations of numerical functions.

This thesis is structured in three chapters we will present briefly below.

The first chapter includes definitions, introductory notions and results needed further development of the results.. Some results and demonstrations are inspired and specified at bibliography, others belong to the author. Also in this chapter are generalized some results known in the literature.

In the second chapter presents the central results of this thesis. Most of the results in this chapter appear in the article: "Some New extensions of classical theorems in number theory", published by the author together with Ph. D. advisor Popa in "Journal of Number Theory" (see [4]). The results in this chapter generalizes a number of famous theorems of analytic theory of numbers, such as the Polya's theorem , as well as a surprising result published of Radoux in 1977. Mention that the results enunciated in article "Jornal of Number Theory" without demonstration we have demonstrated in this thesis. In this thesis we present a number of generalizations of the Radoux result.

The third chapter contains results submitted for possible publication on "International Journal of Number Theory" by the author together with the Ph.D. advisor Dumitru Popa. In this chapter we show a method to obtain asymptotic evaluations for double amounts starting from asymptotic evaluations for simple sums. Using the procedure indicated in the thesis show that all results of the article [4] can be used to obtain asymptotic evaluations for double sums. In this sense we present some results of double Pölya-Riemann and double Riemann-Radoux.

Chapter 1

Classical results in analytical theory of numbers

Next result came as a result of discussion with the Ph.D. advisor, D. Popa. From how many we know does not appear in the literature.

Lemma 1.0.1 *Let* $f : [2, \infty) \to \mathbb{R}$ *be a arbitrary function. Let* $R : [2, \infty) \to \mathbb{R}$ *with the property*

$$R(x) = \sum_{p \leq x} f(p).$$

Then for all $x \in [2, \infty)$ *equality holds*

$$\sum_{p \leq x} \frac{f(p)}{\ln p} = \sum_{k=2}^{x} \frac{R(k) - R(k-1)}{\ln k}.$$

The next result is useful in understanding the demonstrations of theorems and he was suggested by the Ph.D. advisor, D. Popa. From how many we know does not appear in the literature.

Proposition 1.0.1 *Let* $h : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ *o funcţie arbitrară. For all* $x \geq 1$ *equality holds*

$$\sum_{n \leq x} \sum_{d|n} h\left(d, \frac{n}{d}\right) = \sum_{ij \leq x} h\left(i, j\right).$$

Chapter 2

New extensions of some classical theorems in number theory

2.1 Introduction

In the famous paper published in 1896, see [13], J. Hadamard showed that if $\alpha > 1$

$$\lim_{x\to\infty} \frac{1}{x} \sum_{p\leq x,\ p \text{ prime}} \left(\ln \frac{x}{p}\right)^{\alpha-1} \ln p = \Gamma(\alpha) \quad \text{(H)}$$

where Γ is the Euler gamma function and then he used this result in order to prove the prime number theorem.

Few years later, Landau in 1900, see [19], using the prime number theorem, showed that Hadamard's formula (H) is true.

In a classical paper from 1917, see [24], based on the prime number theorem, Polya showed that if $f : [0,1] \to \mathbb{R}$ is a Riemann integrable function on $[0,1]$, then

$$\lim_{x\to\infty} \frac{\ln x}{x} \sum_{p\leq x,\ p \text{ prime}} f\left(\frac{p}{x}\right) = \int_0^1 f(x)\, dx.$$

Some recent applications of Polya's theorem can be found in [7].

In 1977, see [27], Radoux showed that

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \varphi(k) = \frac{6}{\pi^2} \int_0^1 x f(x)\, dx,$$

for all functions $f : [0,1] \to \mathbb{R}$ such that $xf(x)$ is continuous on $[0,1]$, φ is Euler's totient function, see also [16] and [22].

In this chapter we will show that all these apparently different results are in fact of the same kind, see Theorem 2.3.1, Theorem 2.3.5, Theorem 2.3.6, Corollary 2.3.7 and Theorem 2.3.8. Our approach is the following: first, we prove the results for polynomials, then for continuous functions and at the end for Riemann integrable functions.

Let us fix first some notations and notions.

Let $a \in \mathbb{R} \cup \{-\infty\}$, $f : (a, \infty) \to \mathbb{R}$ and $g : (a, \infty) \to \mathbb{R}$ with the property that there exists $b \geq a$ with $g(x) \neq 0$ for all $x \in (b, \infty)$. Throughout the chapter, we will use the following notation: $f(x) \sim g(x)$ as $x \to \infty$ if and only if $\lim\limits_{x \to \infty} \frac{f(x)}{g(x)} = 1$.

We recall that if $g : \mathbb{N} \to [0, \infty)$ is a function, its summatory function $G : (0, \infty) \to [0, \infty)$ is defined by $G(x) = \sum\limits_{n \leq x} g(n)$, see [3, page 39].

If $h : (0, \infty) \to \mathbb{R}$ is such that there exists $x_0 > 0$ with $h(x) \neq 0$ for all $x \geq x_0$ and $g : \mathbb{N} \to [0, \infty)$, we say that the summatory function of g is equivalent to h if and only if $\sum\limits_{n \leq x} g(n) \backsim h(x)$ as $x \to \infty$.

We denote by e the Euler number and we define the sequence $(e_k)_{k \geq 0}$ by $e_0 = 1$, $e_{k+1} = e^{e_k}$ for $k \geq 0$. We also define $\ln_1 x = \ln x = \log_e x$ for $x > 0$ and $\ln_{k+1} x = \ln(\ln_k x)$ for $k \geq 1$ and $x > e_{k-1}$.

Let k be a natural number. We write

$$C\left([0,1]^k\right) := \left\{ f : [0,1]^k \to \mathbb{R} \mid \ f \text{ continuous on } [0,1]^k \right\},$$

which is a real linear space with respect to usual addition and scalar multiplication for functions and a Banach space with respect to the uniform norm i.e. $\|f\|_u = \sup\limits_{(x_1,\dots,x_k) \in [0,1]^k} |f(x_1, \dots, x_k)|$.

Also

$$\begin{aligned} \mathcal{R}\left([0,1]^k\right) &: = \left\{ f : [0,1]^k \to \mathbb{R} \mid \ f \text{ is Riemann integrable on } [0,1]^k \right\} \\ \mathcal{F}[a, \infty) &: = \{ f : [a, \infty) \to \mathbb{R} \}, \ a > 0 \end{aligned}$$

which are real linear spaces with respect to usual addition and scalar multiplication for functions.

If $A \subset \mathbb{R}$ we denote by $\chi_A : \mathbb{R} \to \mathbb{R}$ the characteristic function of A, defined by $\chi_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$.

2.2 Preliminary results

In order to simplify the presentation, we introduce the following definition.

Definition 2.2.1 *We say that a function* $h : (0, \infty) \to \mathbb{R}$ *has the property* P_μ, *with* $\mu \in \mathbb{R}$, *if there exists* $x_0 > 0$ *with* $h(x) > 0$ *for all* $x \geq x_0$, h *is differentiable on* (x_0, ∞) *and* $\lim\limits_{x\to\infty} \frac{xh'(x)}{h(x)} = \mu$.

Proposition 2.2.2 *Let* $f : (0, \infty) \to \mathbb{R}$ *be twice differentiable on* $(0, \infty)$ *and such that there exists* $x_1 > 0$ *with* $f(x) > 0$ *and* $f'(x) \neq 0$ *for all* $x \geq x_1$ *and*

$$\lim_{x\to\infty} \frac{xf'(x)}{f(x)} = C_1 \in \mathbb{R}, \ \lim_{x\to\infty} \frac{xf''(x)}{f'(x)} = C_2 \in \mathbb{R}.$$

Let $h : (0, \infty) \to \mathbb{R}$ *be a function with the property* P_μ *and* $g : \mathbb{N} \to [0, \infty)$ *such that its summatory function is equivalent to* h. *If* $C_1 + \mu > 0$ *and* $C_2 + \mu > -1$, *then* $\lim\limits_{x\to\infty} \frac{1}{h(x)f(x)} \sum\limits_{n\leq x} f(n)\, g(n) = 1 - \frac{C_1}{C_2+\mu+1}$.

Corollary 2.2.3 *Let* $h : (0, \infty) \to \mathbb{R}$ *be a function with the property* P_μ *and* $g : \mathbb{N} \to [0, \infty)$ *such that its summatory function is equivalent to* h. *Also let* $k \in \mathbb{N} \cup \{0\}$. *Then for any* $\alpha_1, ..., \alpha_k \in \mathbb{R}$ *and any* $\alpha_0 > -\mu$, *we have*

$$\lim_{x\to\infty} \frac{1}{x^{\alpha_0} (\ln_1 x)^{\alpha_1} \cdots (\ln_k x)^{\alpha_k} h(x)} \sum_{n\leq x} n^{\alpha_0} (\ln_1 n)^{\alpha_1} \cdots (\ln_k n)^{\alpha_k} g(n) = \frac{\mu}{\alpha_0 + \mu}.$$

The following result, which is of independent interest, shows that under natural hypothesis, in order to prove an asymptotic evaluation for continuous functions, it is enough to prove the same asymptotic evaluation in a dense set.

Theorem 2.2.4 *Let* k *be a natural number,* $a > 0$ *and* $T : C\left([0,1]^k\right) \to \mathcal{F}[a, \infty)$ *a linear operator with the property that there exists* $x_0 \geq a$ *and* $L > 0$ *such that*

$$|(T(f))(x)| \leq L \|f\|_u \ \text{ for all } f \in C\left([0,1]^k\right) \text{ and all } x \geq x_0$$

and $V : C\left([0,1]^k\right) \to \mathbb{R}$ *a bounded linear functional. Further, suppose that there exists* $A \subset C\left([0,1]^k\right)$, A *dense in* $C\left([0,1]^k\right)$, *such that*

$$\lim_{x\to\infty} (T(f))(x) = V(f) \ \text{ for all } f \in A.$$

Then

$$\lim_{x\to\infty} (T(f))(x) = V(f) \ \text{ for all } f \in C\left([0,1]^k\right).$$

The next results use the approximation of Riemann integrable functions with continuous functions, which led to the following result, certainly well known, but for which we could not find a reference in the literature. For the sake of completeness, we shall include here its proof.

Lemma 2.2.1 *Let $f:[0,1]\to\mathbb{R}$ be a Riemann integrable function. Then for any $\varepsilon>0$ there exist two continuous functions $\varphi,\psi:[0,1]\to\mathbb{R}$ such that*

$$\varphi(x)\le f(x)\le\psi(x),\forall x\in[0,1] \text{ and } \int_0^1(\psi(x)-\varphi(x))\,dx\le\varepsilon.$$

2.3 Main results

We begin with a result which can be considered as a multidimensional version of Polya's theorem for continuous functions, see also Theorem 2.3.5, Theorem 2.3.6, Corollary 2.3.7.

Theorem 2.3.1 *Let $h:(0,\infty)\to\mathbb{R}$ be a function with the property P_1 and $g:\mathbb{N}\to[0,\infty)$ such that its summatory function is equivalent to h. Also let $k\in\mathbb{N}\cup\{0\}$. Then for all continuous functions $f:[0,1]^{k+1}\to\mathbb{R}$, we have*

$$\lim_{x\to\infty}\frac{1}{h(x)}\sum_{n\le x}f\left(\frac{n}{x},\frac{\ln_1 n}{\ln_1 x},...,\frac{\ln_k n}{\ln_k x}\right)g(n)=\int_0^1 f(x,\underbrace{1,...,1}_{k\text{-}times})dx.$$

Theorem 2.3.1 suggests the following definition.

Definition 2.3.2 *Let $h:(0,\infty)\to\mathbb{R}$ be a function with the property P_1 and $g:\mathbb{N}\to[0,\infty)$ be such that its summatory function is equivalent to h. For all $k\in\mathbb{N}\cup\{0\}$, we denote by $\mathcal{F}_k\left([0,1]^{k+1},g,h\right)$ the class of all Riemann integrable functions $f:[0,1]^{k+1}\to\mathbb{R}$ with the property that*

$$\lim_{x\to\infty}\frac{1}{h(x)}\sum_{n\le x}f\left(\frac{n}{x},\frac{\ln_1 n}{\ln_1 x},...,\frac{\ln_k n}{\ln_k x}\right)g(n)=\int_0^1 f(x,\underbrace{1,...,1}_{k\text{-}times})dx.$$

From Theorem 2.3.1 we get the following result.

Corollary 2.3.3 *Let $h:(0,\infty)\to\mathbb{R}$ be a function with the property P_1 and $g:\mathbb{N}\to[0,\infty)$ be such that its summatory function is equivalent to h. Then for all $k\in\mathbb{N}\cup\{0\}$, we have*

$$C\left([0,1]^{k+1}\right)\subset\mathcal{F}_k\left([0,1]^{k+1},g,h\right)\subset\mathcal{R}\left([0,1]^{k+1}\right).$$

The following example shows that for $k \in \mathbb{N}$ the inclusion

$$\mathcal{F}_k\left([0,1]^{k+1}, g, h\right) \subset \mathcal{R}\left([0,1]^{k+1}\right)$$

is, in general, strict.

Proposition 2.3.4 *Let $h : (0,\infty) \to \mathbb{R}$ be a function with the property P_1 and*
$g : \mathbb{N} \to [0,\infty)$ be such that its summatory function is equivalent to h. Further suppose that $\lim_{n\to\infty} \frac{g(n)}{h(n)} = 0$. For example $g(n) = 1$, $\forall n \in \mathbb{N}$ and $h(x) = x$, $\forall x > 0$. Let $k \in \mathbb{N}$. Then $f : [0,1]^{k+1} \to \mathbb{R}$ defined by

$$f(x_0, x_1, ..., x_k) = \chi_{\{1\}}(x_1)$$

is Riemann integrable on $[0,1]^{k+1}$ and

$$\lim_{x\to\infty} \frac{1}{h(x)} \sum_{n\le x} f\left(\frac{n}{x}, \frac{\ln_1 n}{\ln_1 x}, ..., \frac{\ln_k n}{\ln_k x}\right) g(n) = 0; \int_0^1 f(x, \underbrace{1, ..., 1}_{k\text{-times}})dx = 1.$$

In the sequel, we give some nontrivial examples of functions that belong to the classes $\mathcal{F}_k\left([0,1]^{k+1}, g, h\right)$.

Theorem 2.3.5 *Let $h : (0,\infty) \to \mathbb{R}$ be a function with the property P_1 and let $g : \mathbb{N} \to [0,\infty)$ be such that its summatory function is equivalent to h. Also let $k \in \mathbb{N} \cup \{0\}$. Then for any Riemann integrable function $\omega : [0,1] \to \mathbb{R}$, any continuous functions $v_1 : [0,1] \to [0,1]$, ..., $v_k : [0,1] \to [0,1]$ with $v_1(1) \cdots v_k(1) \neq 0$, the function $f : [0,1]^{k+1} \to \mathbb{R}$ defined by*

$$f(x_0, x_1, ..., x_k) = \omega\left(x_0 v_1(x_1) \cdots v_k(x_k)\right)$$

belongs to the class $\mathcal{F}_k\left([0,1]^{k+1}, g, h\right)$.

Theorem 2.3.6 *Let $h : (0,\infty) \to \mathbb{R}$ be a function with the property P_1 and let*
$g : \mathbb{N} \to [0,\infty)$ be such that its summatory function is equivalent to h. Also let $k \in \mathbb{N} \cup \{0\}$. If $v_0 : [0,1] \to \mathbb{R}$ is Riemann integrable, $v_1 : [0,1] \to \mathbb{R}$, ..., $v_k : [0,1] \to \mathbb{R}$ are all continuous functions, then the function $f : [0,1]^{k+1} \to \mathbb{R}$ defined by

$$f(x_0, x_1, ..., x_k) = v_0(x_0)\, v_1(x_1) \cdots v_k(x_k)$$

belongs to the class $\mathcal{F}_k\left([0,1]^{k+1}, g, h\right)$.

From Theorem 2.3.5 and Theorem 2.3.6 we have, in particular

Corollary 2.3.7 *Let $h:(0,\infty)\to\mathbb{R}$ be a function with the property P_1 and let*
$g:\mathbb{N}\to[0,\infty)$ be such that its summatory function is equivalent to h. Let $k\in\mathbb{N}\cup\{0\}$. Then:
(i) $\mathcal{F}_0([0,1],g,h)=\mathcal{R}([0,1])$ i.e. for any Riemann integrable function $f:[0,1]\to\mathbb{R}$ we have

$$\lim_{x\to\infty}\frac{1}{h(x)}\sum_{n\leq x} f\left(\frac{n}{x}\right)g(n)=\int_0^1 f(x)\,dx.$$

(ii) If $\omega:[0,1]\to\mathbb{R}$ is Riemann integrable, $v_1:[0,1]\to\mathbb{R}$, ..., $v_k:[0,1]\to\mathbb{R}$ are all continuous functions with $v_1(1)\cdots v_k(1)\neq 0$ we have

$$\lim_{x\to\infty}\frac{1}{h(x)}\sum_{n\leq x}\omega\left(\frac{n}{x}\cdot v_1\left(\frac{\ln_1 n}{\ln_1 x}\right)\cdots v_k\left(\frac{\ln_k n}{\ln_k x}\right)\right)g(n)=\int_0^1\omega(x\cdot v_1(1)\cdots v_k(1))dx.$$

(iii) If $v_0:[0,1]\to\mathbb{R}$ is Riemann integrable, $v_1:[0,1]\to\mathbb{R}$, ..., $v_k:[0,1]\to\mathbb{R}$ are all continuous functions we have

$$\lim_{x\to\infty}\frac{1}{h(x)}\sum_{n\leq x}v_0\left(\frac{n}{x}\right)v_1\left(\frac{\ln_1 n}{\ln_1 x}\right)\cdots v_k\left(\frac{\ln_k n}{\ln_k x}\right)g(n)=v_1(1)\cdots v_k(1)\int_0^1 v_0(x)\,dx.$$

The next result is a new extension of Radoux's theorem.

Theorem 2.3.8 *Let $h:(0,\infty)\to\mathbb{R}$ be a function with the property P_μ, with $\mu\in\mathbb{R}-\{0\}$ and let $g:\mathbb{N}\to[0,\infty)$ be such that its summatory function is equivalent to h. Also let $k\in\mathbb{N}\cup\{0\}$. Then:*
(i) for any $\alpha_1,...,\alpha_k\in\mathbb{R}$, any continuous function $\varphi:[0,1]^{k+1}\to\mathbb{R}$ we have

$$\begin{aligned}&\lim_{x\to\infty}\frac{\sum\limits_{n\leq x}\varphi\left(\frac{n}{x},\frac{\ln_1 n}{\ln_1 x},...,\frac{\ln_k n}{\ln_k x}\right)n^{1-\mu}(\ln_1 n)^{\alpha_1}\cdots(\ln_k n)^{\alpha_k}g(n)}{x^{1-\mu}(\ln_1 x)^{\alpha_1}\cdots(\ln_k x)^{\alpha_k}h(x)}\\ &=\mu\int_0^1\varphi(x,\underbrace{1,...,1}_{k\text{-times}})dx.\end{aligned}$$

(ii) If $\mu\in(1,\infty)$, for any function $f:[0,1]^{k+1}\to\mathbb{R}$ with the property that

$(x_0, x_1, ..., x_k) \to x_0^{\mu-1} f(x_0, x_1, ..., x_k)$ *is continuous on* $[0,1]^{k+1}$ *and any* $\alpha_1, ..., \alpha_k \in \mathbb{R}$ *we have*

$$\begin{aligned} & \lim_{x\to\infty} \frac{\sum\limits_{n\le x} f\left(\frac{n}{x}, \frac{\ln_1 n}{\ln_1 x}, ..., \frac{\ln_k n}{\ln_k x}\right) (\ln_1 n)^{\alpha_1} \cdots (\ln_k n)^{\alpha_k} g(n)}{(\ln_1 x)^{\alpha_1} \cdots (\ln_k x)^{\alpha_k} h(x)} \\ = \ & \mu \int_0^1 x^{\mu-1} f(x, \underbrace{1, ..., 1}_{k\text{-times}}) dx. \end{aligned}$$

(iii) If $\mu \in (1, \infty)$, *for any function* $f : [0,1]^{k+1} \to \mathbb{R}$ *with the property that* $(x_0, x_1, ..., x_k) \to x_1^{\mu-1} f(x_0, x_1, ..., x_k)$ *is continuous on* $[0,1]^{k+1}$ *and any* $\alpha_1, ..., \alpha_k \in \mathbb{R}$ *we have*

$$\begin{aligned} & \lim_{x\to\infty} \frac{\sum\limits_{n\le x} f\left(\frac{n}{x}, \frac{\ln_1 n}{\ln_1 x}, ..., \frac{\ln_k n}{\ln_k x}\right) n^{1-\mu} (\ln_1 n)^{\mu-1} (\ln_1 n)^{\alpha_1} \cdots (\ln_k n)^{\alpha_k} g(n)}{x^{1-\mu} (\ln_1 x)^{\mu-1} (\ln_1 x)^{\alpha_1} \cdots (\ln_k x)^{\alpha_k} h(x)} \\ = \ & \mu \int_0^1 f(x, \underbrace{1, ..., 1}_{k\text{-times}}) dx. \end{aligned}$$

(iv) for any Riemann integrable function $\omega : [0,1] \to \mathbb{R}$, *any continuous functions* $v_1 : [0,1] \to [0,1]$, ..., $v_k : [0,1] \to [0,1]$ *with* $v_1(1) \cdots v_k(1) \neq 0$ *we have*

$$\begin{aligned} & \lim_{x\to\infty} \frac{\sum\limits_{n\le x} \omega\left(\frac{n}{x} \cdot v_1\left(\frac{\ln_1 n}{\ln_1 x}\right) \cdots v_k\left(\frac{\ln_k n}{\ln_k x}\right)\right) n^{1-\mu} (\ln_1 n)^{\alpha_1} \cdots (\ln_k n)^{\alpha_k} g(n)}{x^{1-\mu} (\ln_1 x)^{\alpha_1} \cdots (\ln_k x)^{\alpha_k} h(x)} \\ = \ & \mu \int_0^1 \omega(x \cdot v_1(1) \cdots v_k(1)) dx. \end{aligned}$$

(v) for any Riemann integrable function $v_0 : [0,1] \to \mathbb{R}$, *any continuous functions* $v_1 : [0,1] \to \mathbb{R}$, ..., $v_k : [0,1] \to \mathbb{R}$

$$\begin{aligned} & \lim_{x\to\infty} \frac{\sum\limits_{n\le x} n^{1-\mu} v_0\left(\frac{n}{x}\right) (\ln_1 n)^{\alpha_1} v_1\left(\frac{\ln_1 n}{\ln_1 x}\right) \cdots (\ln_k n)^{\alpha_k} v_k\left(\frac{\ln_k n}{\ln_k x}\right) g(n)}{x^{1-\mu} (\ln_1 x)^{\alpha_1} \cdots (\ln_k x)^{\alpha_k} h(x)} \\ = \ & \mu v_1(1) \cdots v_k(1) \int_0^1 v_0(x) dx. \end{aligned}$$

(vi) If $\mu \in (1, \infty)$, *for any function* $u_0 : [0,1] \to \mathbb{R}$ *such that* $x \to x^{\mu-1} u_0(x)$ *is Riemann integrable, any continuous functions* $v_1 : [0,1] \to \mathbb{R}$, ..., v_k :

$[0,1] \to \mathbb{R}$

$$\lim_{x\to\infty} \frac{\sum\limits_{n\leq x} u_0\left(\frac{n}{x}\right)(\ln_1 n)^{\alpha_1} v_1\left(\frac{\ln_1 n}{\ln_1 x}\right)\cdots(\ln_k n)^{\alpha_k} v_k\left(\frac{\ln_k n}{\ln_k x}\right) g(n)}{(\ln_1 x)^{\alpha_1}\cdots(\ln_k x)^{\alpha_k} h(x)}$$
$$= \mu v_1(1)\cdots v_k(1)\int_0^1 x^{\mu-1}u_0(x)\,dx.$$

Remark 2.3.9 *We leave for the reader to state other possible variants of (ii), (iii) and (vi) in Theorem 2.3.8.*

The following Lemma is a partial answer to a question posed by the referee, see below Question. For this we recall, see [6, 28]:

(i) a function $l:(0,\infty)\to\mathbb{R}$ is called a slowly varying function (in the sense of Karamata) if it is measurable, there exists $x_0>0$ with $l(x)>0$ for all $x\geq x_0$ and $\lim\limits_{x\to\infty}\frac{l(ax)}{l(x)}=1$, $\forall a>0$.

(ii) Let $\mu\in\mathbb{R}$. A function $h:(0,\infty)\to\mathbb{R}$ is called regularly varying with index μ if it is measurable and there exists $l:(0,\infty)\to\mathbb{R}$ a slowly varying function such that $h(x)=x^{\mu}l(x)$, $\forall x>0$.

Lemma 2.3.1 *Let $h:(0,\infty)\to\mathbb{R}$ be a function with the property P_μ, with $\mu\in\mathbb{R}-\{0\}$ and let $g:\mathbb{N}\to[0,\infty)$ be such that its summatory function is equivalent to h. Then h is regularly varying with index μ.*

We leave open the following two problems which were posed by the referee:

Question. a) *Let $g:(0,\infty)\to[0,\infty)$ be a function and G its summatory function and assume that there is a differentiable function $h:(0,\infty)\to\mathbb{R}$ such that $G(x)\backsim h(x)$ as $x\to\infty$ and $\lim\limits_{x\to\infty}\frac{xh'(x)}{h(x)}$ is a real number, different from zero. Is it possible to deduce that g is regularly varying with some index ρ ?*

b) *Let $g:(0,\infty)\to[0,\infty)$ be a slowly varying function (in the sense of Karamata) and G be its summatory function. Can we deduce that there exists a function h such that $G(x)\backsim h(x)$ as $x\to\infty$, where h is differentiable and $\lim\limits_{x\to\infty}\frac{xh'(x)}{h(x)}$ is a real number, different from zero?.*

If the answer to the Question b) is Yes, then Lemma 5.2. from [7] is the consequence of Theorem 2.3.6. If not, then Lemma 5.2. is an independent result. We remark here some interesting comments written in pages 95-96 from [9].

2.4 Some applications

In this section, we apply the above proved results i.e. Theorem 2.3.1, Theorem 2.3.5, Theorem 2.3.6 and Corollary 2.3.7 for some well known arithmetical functions used in number theory, see [1, 3, 17, 20]. We will concentrate only on the case of prime numbers and we leave it for the reader to state and prove other concrete examples of Theorem 2.3.8.

In the sequel we denote by $\mathbb{P}$ the set of all prime numbers and the notation $\sum\limits_{p\le x}\cdots$ means $\sum\limits_{p\le x,\ p \text{ prime}}\cdots$.

Definition 2.4.1 *Let $h:(0,\infty)\to\mathbb{R}$ be a function with the property P_μ. We say that $v:\mathbb{P}\to[0,\infty)$ has its summatory function over the primes equivalent to h if and only if $\sum\limits_{p\le x} v(p)\sim h(x)$ as $x\to\infty$.*

The following result, which establishes a connection between the summatory function over the set of all natural numbers and the summatory function over the primes, is perhaps well known, but we were unable to find it explicitly stated in the literature. Its proof is omitted, since it follows the same lines as in Landau, see [20, page 25], but instead of Cebyshev's estimations, uses the prime number theorem.

Lemma 2.4.1 *Let $f:(0,\infty)\to(0,\infty)$ be such that $\left(\frac{f(n)}{\ln n}\right)_{n\ge 2}$ is a non-increasing sequence and the series $\sum\limits_{n=2}^{\infty}\frac{f(n)}{\ln n}$ is divergent. Then $\sum\limits_{p\le x} f(p)\backsim\sum\limits_{n\le x}\frac{f(n)}{\ln n}$ as $x\to\infty$.*

Definition 2.4.2 *Let $h:(0,\infty)\to\mathbb{R}$ be a function with the property P_μ and $v:\mathbb{P}\to[0,\infty)$ be such that its summatory function over the primes is equivalent to h. Also let $k\in\mathbb{N}\cup\{0\}$. We denote by $\mathcal{F}_k^{prime}\left([0,1]^{k+1},v,h\right)$ the class of all Riemann integrable functions $f:[0,1]^{k+1}\to\mathbb{R}$ with the property that*

$$\lim_{x\to\infty}\frac{1}{h(x)}\sum_{p\le x} f\left(\frac{p}{x},\frac{\ln_1 p}{\ln_1 x},\dots,\frac{\ln_k p}{\ln_k x}\right)v(p)=\int_0^1 f(x,\underbrace{1,\dots,1}_{k\text{-times}})dx.$$

The next result, heuristicly speaking, says that if we know a result for each summatory function over the set of all natural numbers, then we can get a similar result for each summatory function over the primes.

Proposition 2.4.3 *Let $h:(0,\infty)\to\mathbb{R}$ be a function with the property P_μ. Let $k\in\mathbb{N}\cup\{0\}$. If $f\in\mathcal{F}_k\left([0,1]^{k+1},g,h\right)$ for each $g:\mathbb{N}\to[0,\infty)$ such that its summatory function is equivalent to h, then $f\in\mathcal{F}_k^{prime}\left([0,1]^{k+1},v,h\right)$ for each $v:\mathbb{P}\to[0,\infty)$ such that its summatory function over the primes is equivalent to h.*

The following result is a new extension of the classical Polya's theorem. The proof follows from Corollary 2.3.7 and Proposition 2.4.3.

Corollary 2.4.4 *Let $h:(0,\infty)\to\mathbb{R}$ be a function with the property P_1 and $v:\mathbb{P}\to[0,\infty)$ be such that its summatory function over the primes is equivalent to h. Also let $k\in\mathbb{N}\cup\{0\}$. Then:*
(i) $\mathcal{F}_0^{prime}([0,1],g,h)=\mathcal{R}([0,1])$ i.e. for any Riemann integrable function $f:[0,1]\to\mathbb{R}$ we have

$$\lim_{x\to\infty}\frac{1}{h(x)}\sum_{p\le x}f\left(\frac{p}{x}\right)v(p)=\int_0^1 f(x)\,dx.$$

(ii) for any continuous function $f:[0,1]^{k+1}\to\mathbb{R}$

$$\lim_{x\to\infty}\frac{1}{h(x)}\sum_{p\le x}f\left(\frac{p}{x},\frac{\ln_1 p}{\ln_1 x},\ldots,\frac{\ln_k p}{\ln_k x}\right)v(p)=\int_0^1 f(x,\underbrace{1,\ldots,1}_{k\text{-times}})dx.$$

(iii) for any Riemann integrable function $\omega:[0,1]\to\mathbb{R}$, any continuous functions $v_1:[0,1]\to[0,1]$, ..., $v_k:[0,1]\to[0,1]$ with $v_1(1)\cdots v_k(1)\neq 0$

$$\lim_{x\to\infty}\frac{1}{h(x)}\sum_{p\le x}\omega\left(\frac{p}{x}\cdot v_1\left(\frac{\ln_1 p}{\ln_1 x}\right)\cdots v_k\left(\frac{\ln_k p}{\ln_k x}\right)\right)v(p)=\int_0^1\omega(x\cdot v_1(1)\cdots v_k(1))dx.$$

(iv) for any Riemann integrable function $v_0:[0,1]\to\mathbb{R}$, any continuous functions $v_1:[0,1]\to[0,1]$, ..., $v_k:[0,1]\to[0,1]$

$$\lim_{x\to\infty}\frac{1}{h(x)}\sum_{p\le x}v_0\left(\frac{p}{x}\right)v_1\left(\frac{\ln_1 p}{\ln_1 x}\right)\cdots v_k\left(\frac{\ln_k p}{\ln_k x}\right)v(p)=v_1(1)\cdots v_k(1)\int_0^1 v_0(x)dx.$$

In the sequel, we state some particular cases of Corollary 2.4.4. We recall that for $v:\mathbb{P}\to[0,\infty)$, $v(p)=1$, its summatory function over the primes is $\pi:(0,\infty)\to[0,\infty)$, $\pi(x)=\sum\limits_{p\le x}1$ and by the prime number theorem, $\pi(x)\sim\frac{x}{\ln x}$ as $x\to\infty$, see [17, 20].

Also, for $v:\mathbb{P}\to[0,\infty)$, $v(p)=\ln p$, its summatory function over the primes is $\vartheta:(0,\infty)\to[0,\infty)$, $\vartheta(x)=\sum\limits_{p\le x}\ln p$, the first Chebyshev function and by the prime number theorem, $\vartheta(x)\sim x$ as $x\to\infty$, see [17, 20].

Applying Corollary 2.4.4 for these functions we obtain

Corollary 2.4.5 *Let $k \in \mathbb{N} \cup \{0\}$. Then:*
(i) for any continuous function $f : [0,1]^{k+1} \to \mathbb{R}$,

$$\lim_{x\to\infty} \frac{\ln x}{x} \sum_{p\le x} f\left(\frac{p}{x}, \frac{\ln_1 p}{\ln_1 x}, ..., \frac{\ln_k p}{\ln_k x}\right) = \int_0^1 f(x, \underbrace{1, ..., 1}_{k\text{-}times})dx;$$

$$\lim_{x\to\infty} \frac{1}{x} \sum_{p\le x} f\left(\frac{p}{x}, \frac{\ln_1 p}{\ln_1 x}, ..., \frac{\ln_k p}{\ln_k x}\right) \ln p = \int_0^1 f(x, \underbrace{1, ..., 1}_{k\text{-}times})dx.$$

(ii) for any Riemann integrable function $\omega : [0,1] \to \mathbb{R}$ and any continuous functions $v_1 : [0,1] \to [0,1]$, ..., $v_k : [0,1] \to [0,1]$ with $v_1(1) \cdots v_k(1) \neq 0$, we have

$$\lim_{x\to\infty} \frac{\ln x}{x} \sum_{p\le x} \omega\left(\frac{p}{x} \cdot v_1\left(\frac{\ln_1 p}{\ln_1 x}\right) \cdots v_k\left(\frac{\ln_k p}{\ln_k x}\right)\right)$$
$$= \int_0^1 \omega(x \cdot v_1(1) \cdots v_k(1))dx;$$
$$\lim_{x\to\infty} \frac{1}{x} \sum_{p\le x} \omega\left(\frac{p}{x} \cdot v_1\left(\frac{\ln_1 p}{\ln_1 x}\right) \cdots v_k\left(\frac{\ln_k p}{\ln_k x}\right)\right) \ln p$$
$$= \int_0^1 \omega(x \cdot v_1(1) \cdots v_k(1))dx.$$

(iii) for any Riemann integrable function $v_0 : [0,1] \to \mathbb{R}$ and any continuous functions $v_1 : [0,1] \to [0,1]$, ..., $v_k : [0,1] \to [0,1]$,

$$\lim_{x\to\infty} \frac{\ln x}{x} \sum_{p\le x} v_0\left(\frac{p}{x}\right) v_1\left(\frac{\ln_1 p}{\ln_1 x}\right) \cdots v_k\left(\frac{\ln_k p}{\ln_k x}\right)$$
$$= v_1(1) \cdots v_k(1) \int_0^1 v_0(x)dx;$$
$$\lim_{x\to\infty} \frac{1}{x} \sum_{p\le x} v_0\left(\frac{p}{x}\right) v_1\left(\frac{\ln_1 p}{\ln_1 x}\right) \cdots v_k\left(\frac{\ln_k p}{\ln_k x}\right) \ln p$$
$$= v_1(1) \cdots v_k(1) \int_0^1 v_0(x)dx.$$

Chapter 3

Asymptotic evaluations for some double sums in number theory

3.1 Introduction and background

The asymptotic behavior of sequences of sums of various kind is one of fundamental questions in mathematical analysis, number theory and not only. Let us note only that the various and profound results of this type which the reader can found, for example, in the books [1], [3], [20], [31]. In this chapter we indicate a method to obtain asymptotic evaluations for double sums from the asymptotic evaluations from a simple sum, Theorem 3.3.1, Proposition 3.3.2 and Corollary 3.3.3. As applications we show that the all results in the paper [4] can be used to obtain some asymptotic evaluations for double sums, Corolary 3.4.2. By using an old result of Landau's from 1900, we give new double Polya-Riemann type rsults, Corollary 3.4.5, Corollary 3.4.6. Further we give some asymptotic evaluations for double sums which involve the divizor and the Euler totient function, Corollary 3.4.7 and Corollary 3.5.2.

Let us fix first some notation and concepts. Let $a \in \mathbb{R} \cup \{-\infty\}$, $f : (a,\infty) \to \mathbb{R}$ and $g : (a,\infty) \to \mathbb{R}$ with the property that there exists $b \geq a$ with $g(x) \neq 0$ for all $x \in (b,\infty)$. Throughout the paper, we use the notation: $f(x) \sim g(x)$ to mean $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 1$. If $g : \mathbb{N} \to \mathbb{R}$ is a function, its summatory function $G : (0,\infty) \to \mathbb{R}$ is defined by $G(x) = \sum\limits_{n \leq x} g(n)$, see [3, page 39].

If $h : (0,\infty) \to \mathbb{R}$ is such that there exists $x_0 > 0$ with $h(x) \neq 0$ for all $x \geq x_0$ we say that h is eventually non-null. Also if $h : (0,\infty) \to \mathbb{R}$ is eventually non-null and $g : \mathbb{N} \to \mathbb{R}$, we say that the summatory function of g is equivalent to h if $\sum_{n\leq x} g(n) \backsim h(x)$. By $d, \varphi : \mathbb{N} \to \mathbb{N}$ we denote the divizor function repectively the Euler totient function, $d(n) = cardD_n$, $D_n = \{d \in \mathbb{N} \mid d \mid n\}$ and $\varphi(n) = cardP_n$, $P_n = \{a \in \mathbb{N} \mid (a, n) = 1\}$. As is usual $\pi : (0, \infty) \to [0, \infty)$, $\pi(x) = card\{p \mid p \leq x, p \text{ prime}\}$ is the prime number function. Throughout this paper, we use the notation $\sum_{n\leq x}$ to mean $\sum_{n\leq x;\ n\in\mathbb{N}}$, etc. If not otherwise specified all summations are taken over the natural numbers. We denote by e the Euler number and we define the sequence $(e_k)_{k\geq 0}$ by $e_0 = 1$, $e_{k+1} = e^{e_k}$ for $k \geq 0$. We also define $\ln_1 x = \ln x = \log_e x$ for $x > 0$ and $\ln_{k+1} x = \ln(\ln_k x)$ for $k \geq 1$ and $x > e_{k-1}$. As usual if A is a subset of the set X we define the characteristic function $\chi_A : X \to \mathbb{R}$, $\chi_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$.

Definition 3.1.1 *Let $u : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be an arbitrary function. The function $G : (0, \infty) \to \mathbb{R}$ defined by $G(x) = \sum_{ij\leq x} u(i, j)$ is called the double summatory function asociated to u.*
Similar, if $A, B \subseteq \mathbb{N}$ and $v : A \times B \to \mathbb{R}$ is an arbitrary function, we define its double summatory function $G_{A,B} : (0, \infty) \to \mathbb{R}$ by $G_{A,B}(x) = \sum_{ij\leq x,\ (i,j)\in A\times B} v(i, j)$.

If $h : (0, \infty) \to \mathbb{R}$ is eventually non-null, $A, B \subseteq \mathbb{N}$ and $v : A \times B \to \mathbb{R}$, we say that the double summatory function of v is equivalent to h if $\sum_{ij\leq x,\ (i,j)\in A\times B} v(i, j) \backsim h(x)$.

3.2 Preliminary results

One of the main ingredient in our proofs will be the following almost obvious result. It will gives us the posibility to use the results known for a "single sum" to obtain the results for "double sum".

Proposition 3.2.1 *Let $u : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be an arbitrary function. For all $x \geq 1$ the following equality holds*

$$\sum_{ij\leq x} u(i, j) = \sum_{n\leq x} \sum_{d|n} u\left(d, \frac{n}{d}\right).$$

In other words, if $g : \mathbb{N} \to \mathbb{R}$ is defined by $g(n) = \sum_{d|n} u\left(d, \frac{n}{d}\right)$, then the double summatory function of u is equal to the summatory function of g.

Corollary 3.2.2 *(a) Let $f:\mathbb{N}\to\mathbb{R}$ be an arbitrary function. For all $x\geq 1$ the following equality holds* $\sum\limits_{n\leq x} f(n)\,d(n)=\sum\limits_{ij\leq x} f(ij)$.
(b) Let $f,g:\mathbb{N}\to\mathbb{R}$ be two arbitrary functions. For all $x\geq 1$ the following equality holds $\sum\limits_{n\leq x}\sum\limits_{d|n} f(d)\,g\left(\frac{n}{d}\right)=\sum\limits_{ij\leq x} f(i)\,g(j)$*; in other words, if $f*g$ is the Dirichlet convolution of f and g, then* $\sum\limits_{n\leq x}(f*g)(n)=\sum\limits_{ij\leq x} f(i)\,g(j)$.

Let us note that the item (b) in Corollary 3.2.2 is almost implicit in every book of number theory, see for example [31], page 37, where the author write: "The left-hand side of (2) can be written..."

In the sequel we prove the asymptotic evaluations for some double summatory functions associated to the divizor function and the Euler totient function.

Proposition 3.2.3 *The following evaluation holds:*
$\sum\limits_{n\leq x}\frac{d(n)\ln n}{n}=\frac{1}{3}\ln^3 x+C\ln^2 x+O(\ln x)$, *$C$ is the Euler constant.*

Proposition 3.2.4 *The following evaluation holds:*
(i) $\sum\limits_{ij\leq x}\varphi(i)\varphi(j)=\frac{18x^2\ln x}{\pi^4}+\frac{\left(36C-6A\pi^2-9\right)x^2}{\pi^4}+O(x\sqrt{x}\ln x)$, *$C$ is the Euler constant and* $A=\sum\limits_{n=1}^{\infty}\frac{\mu(n)\ln n}{n^2}$.
(ii) $\sum\limits_{ij\leq x} d(i)\,d(j)=\frac{x\ln^3 x}{6}+\frac{(4C-1)x\ln^2 x}{2}+O(x\ln x)$, *$C$ is the Euler constant.*
(iii) $\sum\limits_{ij\leq x} d(i)\,\varphi(j)=\frac{\pi^2x^2}{12}+O(x\sqrt{x}\ln x)$.

3.3 The main results

The next result show that if we know an asymptotic evaluation for a "single variable" we can deduce an asymptotic result for "double variable".

Theorem 3.3.1 *Let $h:(0,\infty)\to\mathbb{R}$ be an eventually non-null and let $k\in\mathbb{N}\cup\{0\}$. If $f:[0,1]^{k+1}\to\mathbb{R}$ is a function with the property that there exist*

$$\lim_{x\to\infty}\frac{1}{h(x)}\sum_{n\leq x} f\left(\frac{n}{x},\frac{\ln_1 n}{\ln_1 x},...,\frac{\ln_k n}{\ln_k x}\right)g(n)=L(f)\in\mathbb{R}$$

for all $g:\mathbb{N}\to\mathbb{R}$ such that its summatory function is equivalent to h, where $L(f)$ depend only on f, then

$$\lim_{x\to\infty}\frac{1}{h(x)}\sum_{ij\leq x} f\left(\frac{ij}{x},\frac{\ln_1(ij)}{\ln_1 x},...,\frac{\ln_k(ij)}{\ln_k x}\right)u(i,j)=L(f)$$

for all $u : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ *such that its double summatory function is equivalent to* h*.*

The next result, heuristically speaking, says that if we know a result for each double summatory function on $\mathbb{N} \times \mathbb{N}$, then we can deduce a similar result for each double summatory function over the cartesian product of two subsets of the set of all natural numbers.

Proposition 3.3.2 *Let* $h : (0, \infty) \to \mathbb{R}$ *be an eventually non-null and let* $k \in \mathbb{N} \cup \{0\}$*. If* $f : [0,1]^{k+1} \to \mathbb{R}$ *is a function with the property that there exist*

$$\lim_{x \to \infty} \frac{1}{h(x)} \sum_{ij \le x} f\left(\frac{ij}{x}, \frac{\ln_1(ij)}{\ln_1 x}, ..., \frac{\ln_k(ij)}{\ln_k x}\right) u(i,j) = L(f) \in \mathbb{R}$$

for all $u : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ *such that its double summatory function is equivalent to* h*, where* $L(f)$ *depend only on* f*, then*

$$\lim_{x \to \infty} \frac{1}{h(x)} \sum_{ij \le x, \ (i,j) \in A \times B} f\left(\frac{ij}{x}, \frac{\ln_1(ij)}{\ln_1 x}, ..., \frac{\ln_k(ij)}{\ln_k x}\right) v(i,j) = L(f)$$

for all $A, B \subseteq \mathbb{N}$ *and all* $v : A \times B \to \mathbb{R}$ *such that its double summatory function is equivalent to* h*.*

From Theorem 3.3.1 and Proposition 3.3.2 de deduce

Corollary 3.3.3 *Let* $h : (0, \infty) \to \mathbb{R}$ *be an eventually non-null and let* $k \in \mathbb{N} \cup \{0\}$*. If* $f : [0,1]^{k+1} \to \mathbb{R}$ *is a function with the property that there exist*

$$\lim_{x \to \infty} \frac{1}{h(x)} \sum_{n \le x} f\left(\frac{n}{x}, \frac{\ln_1 n}{\ln_1 x}, ..., \frac{\ln_k n}{\ln_k x}\right) g(n) = L(f) \in \mathbb{R}$$

for all $g : \mathbb{N} \to \mathbb{R}$ *such that its summatory function is equivalent to* h*, where* $L(f)$ *depend only on* f*, then*

$$\lim_{x \to \infty} \frac{1}{h(x)} \sum_{ij \le x, \ (i,j) \in A \times B} f\left(\frac{ij}{x}, \frac{\ln_1(ij)}{\ln_1 x}, ..., \frac{\ln_k(ij)}{\ln_k x}\right) v(i,j) = L(f)$$

for all $A, B \subseteq \mathbb{N}$ *and all* $v : A \times B \to \mathbb{R}$ *such that its double summatory function is equivalent to* h*.*

3.4 Double Polya-Riemann type results

In the sequel we will prove some results which for obvious reasons we call double Polya-Riemann type results, see [4], [24]. For this we recall the following definition, see [4, Definition 1].

Definition 3.4.1 *We say that a function* $h : (0, \infty) \to \mathbb{R}$ *has the property* P_μ*, with* $\mu \in \mathbb{R}$*, if there exists* $x_0 > 0$ *with* $h(x) > 0$ *for all* $x \geq x_0$*,* h *is differentiable on* (x_0, ∞) *and* $\lim\limits_{x\to\infty} \frac{xh'(x)}{h(x)} = \mu$.

Corollary 3.4.2 *Let* $h : (0, \infty) \to \mathbb{R}$ *be a function with the property* P_1*. Let also* $k \in \mathbb{N} \cup \{0\}$*,* $A, B \subseteq \mathbb{N}$ *and* $v : A \times B \to [0, \infty)$ *be such that its double summatory function is equivalent to* h*. Then:*
(i) for any Riemann integrable function $f : [0, 1] \to \mathbb{R}$ *the following equality holds*

$$\lim_{x\to\infty} \frac{1}{h(x)} \sum_{ij\leq x,\ (i,j)\in A\times B} f\left(\frac{ij}{x}\right) v(i,j) = \int_0^1 f(x)\,dx.$$

(ii) If $\omega : [0, 1] \to \mathbb{R}$ *is Riemann integrable,* $v_1 : [0, 1] \to [0, 1]$*, ...,* $v_k : [0, 1] \to [0, 1]$ *are all continuous functions with* $v_1(1)\cdots v_k(1) \neq 0$ *the following equality holds*

$$\begin{aligned} &\lim_{x\to\infty} \frac{1}{h(x)} \sum_{ij\leq x,\ (i,j)\in A\times B} \omega\left(\frac{ij}{x} \cdot v_1\left(\frac{\ln_1(ij)}{\ln_1 x}\right) \cdots v_k\left(\frac{\ln_k(ij)}{\ln_k x}\right)\right) v(i,j) \\ = &\int_0^1 \omega(x \cdot v_1(1) \cdots v_k(1))dx. \end{aligned}$$

(iii) If $v_0 : [0, 1] \to \mathbb{R}$ *is Riemann integrable,* $v_1 : [0, 1] \to \mathbb{R}$*, ...,* $v_k : [0, 1] \to \mathbb{R}$ *are all continuous functions the following equality holds*

$$\begin{aligned} &\lim_{x\to\infty} \frac{1}{h(x)} \sum_{ij\leq x,\ (i,j)\in A\times B} v_0\left(\frac{ij}{x}\right) v_1\left(\frac{\ln_1(ij)}{\ln_1 x}\right) \cdots v_k\left(\frac{\ln_k(ij)}{\ln_k x}\right) v(i,j) \\ = \ &v_1(1) \cdots v_k(1) \int_0^1 v_0(x)\,dx. \end{aligned}$$

In the sequel we give non-trivial examples of functions for which we find the asymptotic evaluations of its double summatory functions. For this we need the following result of E. Landau from 1900, see [19, page 28] or [20, pages 203-205]. We note that Landau's theorem use the prime number theorem.

Theorem 3.4.3 *Let* $F:(0,\infty)\times(0,\infty)\to\mathbb{R}$ *be such that* $F(\nu,x)\geq 0$ *for* $1\leq\nu\leq x$; $\frac{F(\nu,x)}{\ln\nu}\geq\frac{F(\nu',x)}{\ln\nu'}$ *for* $2\leq\nu\leq\nu'\leq x$ *and* $F(2,x)=o\left(\int_2^x\frac{F(u,x)}{\ln u}du\right)$. *Then* $\sum\limits_{p\leq x}F(p,x)\backsim\int_2^x\frac{F(u,x)}{\ln u}du$.

Proposition 3.4.4 *The following evaluations holds:*
(i) card$(\{(i,j)\in\mathbb{N}\times\mathbb{N}\mid ij\leq x\})\backsim x\ln x$.
(ii) card$\{(p,j)\in\mathbb{N}\times\mathbb{N}\mid pj\leq x,\ p\ \text{prime}\}\backsim x\ln(\ln x)$.
(iii) card$\{(p,q)\in\mathbb{N}\times\mathbb{N}\mid pq\leq x,\ p\ \text{prime},\ q\ \text{prime}\}\backsim\frac{2x\ln(\ln x)}{\ln x}$.

From Corollary 3.4.2 and Proposition 3.4.4 we deduce

Corollary 3.4.5 *For any Riemann integrable function* $f:[0,1]\to\mathbb{R}$ *the following equalities holds*

$$\begin{aligned}
\lim_{x\to\infty}\frac{1}{x\ln x}\sum_{ij\leq x,\ i\ \text{natural},\ j\ \text{natural}} f\left(\frac{ij}{x}\right) &= \int_0^1 f(x)\,dx;\\
\lim_{x\to\infty}\frac{1}{x\ln(\ln x)}\sum_{pj\leq x,\ p\ \text{prime},\ j\ \text{natural}} f\left(\frac{pj}{x}\right) &= \int_0^1 f(x)\,dx;\\
\lim_{x\to\infty}\frac{\ln x}{x\ln(\ln x)}\sum_{pq\leq x,\ p\ \text{prime},\ q\ \text{prime}} f\left(\frac{pq}{x}\right) &= 2\int_0^1 f(x)\,dx.
\end{aligned}$$

Corollary 3.4.6 *Let* $k\in\mathbb{N}\cup\{0\}$.
(i) If $\omega:[0,1]\to\mathbb{R}$ *is Riemann integrable,* $v_1:[0,1]\to[0,1]$, ..., $v_k:[0,1]\to[0,1]$ *are all continuous functions with* $v_1(1)\cdots v_k(1)\neq 0$, *then the following equalities holds*

$$\begin{aligned}
&\lim_{x\to\infty}\frac{1}{x\ln x}\sum_{ij\leq x,\ i\ \text{natural},\ j\ \text{natural}}\omega\left(\frac{ij}{x}\cdot v_1\left(\frac{\ln_1(ij)}{\ln_1 x}\right)\cdots v_k\left(\frac{\ln_k(ij)}{\ln_k x}\right)\right)\\
=&\int_0^1\omega(x\cdot v_1(1)\cdots v_k(1))dx;\\
&\lim_{x\to\infty}\frac{1}{x\ln(\ln x)}\sum_{pj\leq x,\ p\ \text{prime},\ j\ \text{natural}}\omega\left(\frac{pj}{x}\cdot v_1\left(\frac{\ln_1(pj)}{\ln_1 x}\right)\cdots v_k\left(\frac{\ln_k(pj)}{\ln_k x}\right)\right)\\
=&\int_0^1\omega(x\cdot v_1(1)\cdots v_k(1))dx;\\
&\lim_{x\to\infty}\frac{\ln x}{x\ln(\ln x)}\sum_{pq\leq x,\ p\ \text{prime},\ q\ \text{prime}}\omega\left(\frac{pq}{x}\cdot v_1\left(\frac{\ln_1(pq)}{\ln_1 x}\right)\cdots v_k\left(\frac{\ln_k(pq)}{\ln_k x}\right)\right)\\
=&\ 2\int_0^1\omega(x\cdot v_1(1)\cdots v_k(1))dx.
\end{aligned}$$

(ii) If $v_0 : [0,1] \to \mathbb{R}$ *is Riemann integrable,* $v_1 : [0,1] \to \mathbb{R}$, ..., $v_k : [0,1] \to \mathbb{R}$ *are all continuous functions, then the following equalities holds*

$$\lim_{x\to\infty} \frac{1}{x \ln x} \sum_{ij\leq x,\ i \text{ natural},\ j \text{ natural}} v_0\left(\frac{ij}{x}\right) v_1\left(\frac{\ln_1 (ij)}{\ln_1 x}\right) \cdots v_k\left(\frac{\ln_k (ij)}{\ln_k x}\right)$$

$$= v_1(1) \cdots v_k(1) \int_0^1 v_0(x)\, dx;$$

$$\lim_{x\to\infty} \frac{1}{x \ln(\ln x)} \sum_{pj\leq x,\ p \text{ prime},\ j \text{ natural}} v_0\left(\frac{pj}{x}\right) v_1\left(\frac{\ln_1 (pj)}{\ln_1 x}\right) \cdots v_k\left(\frac{\ln_k (pj)}{\ln_k x}\right)$$

$$= v_1(1) \cdots v_k(1) \int_0^1 v_0(x)\, dx;$$

$$\lim_{x\to\infty} \frac{\ln x}{x \ln(\ln x)} \sum_{pq\leq x,\ p \text{ prime},\ q \text{ prime}} v_0\left(\frac{pq}{x}\right) v_1\left(\frac{\ln_1 (pq)}{\ln_1 x}\right) \cdots v_k\left(\frac{\ln_k (pq)}{\ln_k x}\right)$$

$$= 2v_1(1) \cdots v_k(1) \int_0^1 v_0(x)\, dx.$$

The next result is of different type than those proved above.

Corollary 3.4.7 *Let* $k \in \mathbb{N} \cup \{0\}$. *Then:*
(i) for any Riemann integrable function $f : [0,1] \to \mathbb{R}$ *the following equality holds*

$$\lim_{x\to\infty} \frac{1}{x \ln^3 x} \sum_{ij\leq x} f\left(\frac{ij}{x}\right) d(i)\, d(j) = \frac{1}{6} \int_0^1 f(x)\, dx.$$

(ii) If $\omega : [0,1] \to \mathbb{R}$ *is Riemann integrable,* $v_1 : [0,1] \to [0,1]$, ..., $v_k : [0,1] \to [0,1]$ *are all continuous functions with* $v_1(1) \cdots v_k(1) \neq 0$ *the following equality holds*

$$\lim_{x\to\infty} \frac{1}{x \ln^3 x} \sum_{ij\leq x} \omega\left(\frac{ij}{x} \cdot v_1\left(\frac{\ln_1 (ij)}{\ln_1 x}\right) \cdots v_k\left(\frac{\ln_k (ij)}{\ln_k x}\right)\right) d(i)\, d(j)$$

$$= \frac{1}{6} \int_0^1 \omega(x \cdot v_1(1) \cdots v_k(1)) dx.$$

(iii) If $v_0 : [0,1] \to \mathbb{R}$ *is Riemann integrable,* $v_1 : [0,1] \to \mathbb{R}$, ..., $v_k : [0,1] \to \mathbb{R}$ *are all continuous functions the following equality holds*

$$\lim_{x\to\infty} \frac{1}{x \ln^3 x} \sum_{ij\leq x} v_0\left(\frac{ij}{x}\right) v_1\left(\frac{\ln_1 (ij)}{\ln_1 x}\right) \cdots v_k\left(\frac{\ln_k (ij)}{\ln_k x}\right) d(i)\, d(j)$$

$$= \frac{1}{6} v_1(1) \cdots v_k(1) \int_0^1 v_0(x)\, dx.$$

3.5 Double Riemann-Radoux type results

In the sequel we prove what we call double Riemann-Radoux type results, see [4], [27]. This time instead to state the general posible statements we mainly concentrate on some particular casses and leave for the reader to state the general statements. From Corollary 3.3.3 and Theorem 5 in [4] we deduce

Corollary 3.5.1 *Let $h:(0,\infty)\to\mathbb{R}$ be a function with the property P_μ, $\mu\in\mathbb{R}-\{0\}$, $A,B\subseteq\mathbb{N}$ and $v:A\times B\to[0,\infty)$ such that its double summatory function is equivalent to h. Then:*
(i) for any Riemann integrable function $f:[0,1]\to\mathbb{R}$ the following equality holds

$$\lim_{x\to\infty}\frac{\sum\limits_{ij\le x,\ (i,j)\in A\times B} f\left(\frac{ij}{x}\right)(ij)^{1-\mu}v(i,j)}{x^{1-\mu}h(x)}=\mu\int_0^1 f(x)\,dx.$$

(ii) If $\mu\in(1,\infty)$, for any function $f:[0,1]\to\mathbb{R}$ such that $x\to x^{\mu-1}f(x)$ is Riemann integrable the following equality holds

$$\lim_{x\to\infty}\frac{\sum\limits_{ij\le x,\ (i,j)\in A\times B} f\left(\frac{ij}{x}\right)v(i,j)}{h(x)}=\mu\int_0^1 x^{\mu-1}f(x)\,dx.$$

Next we prove some concrete examples of Corollary 3.5.1.

Corollary 3.5.2 *For any function $f:[0,1]\to\mathbb{R}$ such that $x\to xf(x)$ is Riemann integrable, the following equalities holds*

$$\lim_{x\to\infty}\frac{\sum\limits_{ij\le x} f\left(\frac{ij}{x}\right)\varphi(i)\varphi(j)}{x^2\ln x}=\frac{36}{\pi^4}\int_0^1 xf(x)\,dx.$$

$$\lim_{x\to\infty}\frac{\sum\limits_{ij\le x} f\left(\frac{ij}{x}\right)d(i)\,\varphi(j)}{x^2}=\frac{\pi^2}{6}\int_0^1 xf(x)\,dx.$$

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